KNOT THEORY NOTES

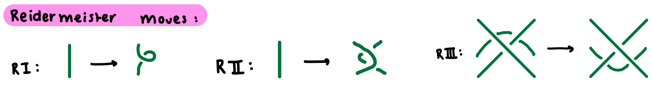
0] Foundations

Dfn: an oriented knot in $\mathbb{R}^3$ is an isotopy class of embeddings $K : S^1 \hookrightarrow \mathbb{R}^3$.

Dfn: a knot diagram is a smooth map $\gamma : S^1 \to \mathbb{R}^2$ s.t $\gamma'(p) \neq 0 \ \forall p \in S^1$, if $\gamma(p) = \gamma(p')$, then the intersection is transverse, and we have an ordering, and no triple intersection.

Thm: almost all projections of knots $\subset \mathbb{R}^3 \to \mathbb{R}^2$ are knot diagrams.

Reidermeister moves:

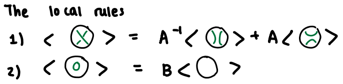


1] Jones Polynomial

Prop: There is a ! map $\langle \ \rangle : \{\text{knot diagrams}\} \to \mathbb{Z}[A^{\pm 1}, B]$ satisfying

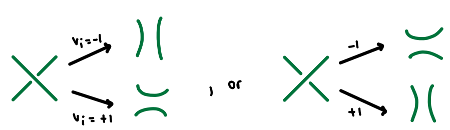
0) $\langle \emptyset \rangle = 1$

The local rules



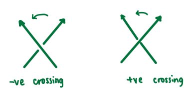
set $B = -A^2 - A^{-2}$

Equivalently,



Rem: $\langle \ \rangle$ is invariant under RII and RIII when we set $B = -A^{-2} - A^2$. But RI is never: $\langle \text{ } \rangle = -A^{-3} \langle | \rangle$.

If D is an oriented link diagram, then every crossing looks like



let $n_\pm(D) :=$ # of $\pm$ crossings. The writhe of D is $w(D) = n_+(D) - n_-(D)$.

Thm: If D is a link diagram, then $\bar{V}(D) := (-A^3)^{-w(D)} \langle D \rangle$ is invariant under Reidermeister moves.

Dfn: if L is an oriented link,

$$\bar{V}(L) := (-A^3)^{-w(D)} \langle D \rangle,$$

where D is any diagram of L, is the unnormalized Jones polynomial of V.

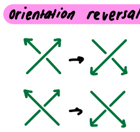
Corollary: If D is a diagram of the unknot, then $\langle D \rangle = (-A^3)^{w(D)} B$.

Better normalization The normalized Jones polynomial of L is

$$V_L(q) = \frac{\bar{V}(L)}{B} = \frac{\bar{V}(L)}{\bar{V}(\circ)} \Big|_{q = -A^{-2}}$$ ← divide by $-A^2 - A^{-2}$

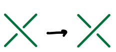
Operations on knots and links:

orientation reversal: (reverse on all components)



$\Rightarrow \begin{cases} \langle r(D) \rangle = \langle D \rangle, & w(r(D)) = w(D) \\ \bar{V}(r(L)) = \bar{V}(L) \text{ and } & V(r(L)) = V(L) \end{cases}$

Mirror:



$\Rightarrow \begin{cases} \langle \bar{D} \rangle = \langle D \rangle |_{A \mapsto A^{-1}}, & w(\bar{D}) = -w(D) \\ \bar{V}(\bar{L}) = \bar{V}(L)|_{A \mapsto A^{-1}}, & V(\bar{L}) = V(L)|_{q \mapsto q^{-1}} \end{cases}$

Disjoint union: $\begin{cases} \langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle, \ w(D_1 \sqcup D_2) = w(D_1) + w(D_2) \\ \bar{V}(L_1 \sqcup L_2) = \bar{V}(L_1)\bar{V}(L_2), \ V(L_1 \sqcup L_2) = V(L_1)V(L_2)\ V(\bar{0})|_{q = -A^{-2}} \end{cases}$

Connected sum: $V(K_1 \# K_2) = V(K_1)V(K_2)$

Oriented Skein relation

$\Rightarrow V(L) \in \mathbb{Z}[q^{\pm}]$, $|L| = \text{odd} \Rightarrow$ even powers, $|L| = \text{even} \Rightarrow$ odd powers $\qquad V_L(1) = 2^{|L|-1}$

Crossing number

$$\langle D \rangle = \sum_{v \in \{\pm 1\}^n} A^{\Sigma v_i} B^{|D_v|} = \sum_v \langle D \rangle_v \quad \text{where } \langle D \rangle_v := A^{\Sigma v_i} B^{|D_v|}$$

Let
- $M(D)$ = maximum power of A in $\langle D \rangle$
- $m(D)$ = minimum power of A in $\langle D \rangle$
- $M_v(D)$ = maximum power of A in $\langle D \rangle_v = \Sigma v_i + 2|D_v|$
- $m_v(D)$ = minimum power of A in $\langle D \rangle_v = \Sigma v_i - 2|D_v|$.

Lemma: if D is a connected planar diagram with n crossings, $|D_{v+}| + |D_{v-}| \leq n + 2$

Colorings, Alternating diagrams

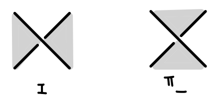
$c(D) =$ # crossings in D

Definition: say L is nonsplit if every diagram D representing L is connected. That is, $L \neq L_1 \sqcup L_2$ for L_1, L_2 nonempty.

Dfn: a diagram D is alternating if crossings alternate between over and under

Lemma: every planar diagram has exactly 2 checkerboard colorings form two new planar graphs $B(D)$, $W(D)$.

At a crossing: 2 possibilities

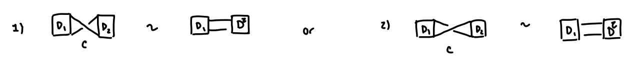


We'll say that a coloring is consistent if all crossings are I or all crossings are II.

Lemma: if D is a connected planar diagram, then D is consistent iff D is alternating.

Lemma: if D is a connected alternating diagram, then $|D_{v-}| + |D_{v+}| = c(D) + 2$

Dfn: A crossing c of D is nugatory if D looks like



We say D is reduced if it has no nugatory crossing

Lemma: if D is a reduced, alternating diagram,
then $m(D) = m(\langle D \rangle_{v-}) = -n - 2|D_{v-}|$
and $M(D) = M(\langle D \rangle_{v+}) = n + 2|D_{v+}|$.

Corollary: if D is a reduced alternating diagram, then $M(D) - m(D) \leq 4n + 4$

Definition: A connected planar graph G is small if every edge is either a loop or a bridge

Proposition: If G is small, then
a) G has a unique maximal tree
b) If D is any planar diagram, with $B(D) = G$, then D can be unknotted using only RI moves (is the unknot)

Definition: let $\mathcal{U}(G) = \{ T \subset G : T \text{ is a maximal tree} \}$

Corollary: if $V(L) = q^k \left(\Sigma a_i q^{2i} \right)$, then $\Sigma |a_i| \leq \# \mathcal{U}(G)$

Theorem: if D is a connected alternating diagram, then
a) $V(L)$ is alternating and
b) $\Sigma |a_i| = | V(L)|_{q^2 = -1} | = \# \mathcal{U}(G)$

Definition: if L is a link, it's determinant is $\det(L) := |V(L)|_{q^2 = -1}|$.

L split $\Rightarrow \det(L) = 0$.

if L is alternating, $\det(L) = \# \mathcal{U}(G)$

2] Alexander Polynomial

Definition: the exterior of a compact submanifold $N \subset M$ is $M \setminus j(D^\circ(\upsilon))$, where $j(D^\circ(\upsilon))$ is the interior of a tubular neighbourhood of N in M

Lemma: if $K: S^1 \hookrightarrow S^3$, then $\upsilon_{S^3/K}$ is trivial.

Proposition: suppose $L: \hat{\sqcup} S^1 \hookrightarrow S^3$ is a link. Then

$$H_*(E_L) = \begin{cases} \mathbb{Z}^{n-1} & \text{if } * = 2 \\ \mathbb{Z}^n & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \end{cases}$$

and $H_1(E_L) = \langle m_1, \dots, m_n \rangle$ where m_i is the meridian of the i^{th} component

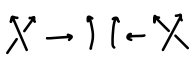
Proposition:
1) $H_1(T^2) \cong \mathbb{Z}^2$
2) if $\alpha \in H_1(T^2)$ $\alpha \in [\gamma]$, where $\gamma: S^1 \hookrightarrow T^2$ is a simple closed curve iff α is primitive (i.e. $\alpha \neq k\beta$ for some $k > 1$),
3) if $\gamma, \gamma' : S^1 \hookrightarrow T^2$ with $[\gamma] = [\gamma']$, then $\gamma \sim_i \gamma'$.

Seifert Surfaces

Theorem: every $K \hookrightarrow S^3$ has a seifert surface

Seifert's algorithm: given a diagram D of K, construct a seifert surface

1. Give every crossing the oriented resolution:



2. Resulting diagram has no crossings, and a natural orientation coming from the orientation on D.

3. Let c be a circle of the resulting diagram. It bounds a disk $D_c \subset \mathbb{R}^2$.
Let n_c = mod 2 # of circles in the resolved diagram that separate c from ∞.
Let $r_c = 0$ if c is oriented counter clockwise, and $r_c = 1$ if c is oriented clockwise.
Orient D_c according to sum $n_c + r_c$:
Color D_c red if $n_c + r_c$ is odd, and blue if $n_c + r_c$ is even.

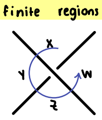
4. Attach a band  of surface at each crossing.

Euler characteristic of Seifert surface: χ = # circles − # crossings
(detract to graph)

From homology: $\chi = 1 - 2g$, g = genus

Fundamental Group of Link Exterior

Dehn presentation: Generators are finite regions of $\mathbb{R}^2 \setminus D_{graph}$.
Relations $\leftrightarrow$ crossings :

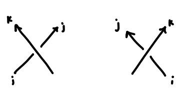


$\longleftrightarrow$ $xy^{-1}zw^{-1} = 1$

infinite region corresponds to $1 \in \pi_1(E_L)$

Wirtinger Presentation:

Let D be an oriented planar diagram. An arc of D is part of D that I can draw without lifting up the chalk.
If there are n crossings, then there are n arcs. The group G_{wirt} has generators $\gamma_1, \dots, \gamma_n \leftrightarrow$ arcs and relations $w_1, \dots, w_n$ corresp. to crossings.



$\gamma_j = \gamma_k \gamma_i \gamma_k^{-1}$ $\qquad$ $\gamma_j = \gamma_k^{-1} \gamma_i \gamma_k$

Definition: Suppose $N_0, N_1 \subset M$ are smooth submanifolds. N_0 and N_1 are ambient isotopic if theres a diffeomorphism $f: M \to M$ such that
1) $f(N_0) = N_1$
2) $f \sim_i id_M$

$\Rightarrow f: M \setminus N_0 \xrightarrow{\sim} M \setminus N_1$

Theorem: (compact) isotopy $\Rightarrow$ ambient isotopy

Corollary: L_0, L_1 isotopic and j_0, j_1 tubular nhoods, then $S^3 \setminus Im(j_0) \cong S^3 \setminus Im(j_1)$

Alexander Polynomial

abelianization map: $|\cdot| : \pi_1(E_K) \to H_1(E_K) \cong \mathbb{Z}$. $\exists$ connected covering space $p: \tilde{E}_K \to E_K$ with $\ker|\cdot| = \pi_1(\tilde{E}_K)$. $G_{deck} = \pi_1(E_K)/\ker|\cdot| \cong \mathbb{Z}$

Definition: $\tilde{E}_K$ is the infinite cyclic cover of E_K.

Definition: the Alexander module of K is $A(K) = H_1(\tilde{E}_K)$ as a module over $R = \mathbb{Z}[H_1(E_K)] = \mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t^{\pm 1}]$ where $t \cdot x = \varphi_*(x)$.

$\pi_1(E_K) = \pi_1(\bar{E}_K)$, $A(K) \simeq A(\bar{K})$.

Theorem (Gordon + Leuke):
if E_K is orientation preserving homeomorphic to $E_{K'}$, then $K \sim_i K'$

Structure Theorem for finitely generated modules over a P.I.D: $\qquad R_{\mathbb{Q}} = \mathbb{Q}[t^{\pm 1}]$

$$\Rightarrow H_1(\tilde{E}_K; \mathbb{Q}) \simeq \underbrace{R_{\mathbb{Q}}^k}_{\text{free}} \oplus (\underbrace{R_{\mathbb{Q}}/p_1 \oplus \cdots \oplus R_{\mathbb{Q}}/p_r}_{\text{Torsion}})$$

Lemma: $H_1(\tilde{E}_K; \mathbb{Q})$ is a Torsion module over $R_{\mathbb{Q}}$.

$$C_*^{cell}(X_K) \simeq C_*^{cell}(\tilde{X}_K) \otimes_{R_{\mathbb{Q}}} M_{t-1} \quad \text{where} \quad M_{t-1} = R_{\mathbb{Q}}/(t-1).$$

A consequence: $A(K; \mathbb{Q}) \simeq R_{\mathbb{Q}}/p_1 \oplus \dots \oplus R_{\mathbb{Q}}/p_r$ is a Torsion module.

Define the Alexander polynomial $\Delta_K(t) = \prod_{i=1}^{r} p_i \in R_{\mathbb{Q}}$ to be the "order" of $A(K)$. This is well-defined up to $\times$ by units in $R_{\mathbb{Q}}$, i.e. up to multiplication by ct^i, where $c \in \mathbb{Q}$, $i \in \mathbb{Z}$.

Example:
- $\Delta_u(t) \sim 1$ (take $R/\langle 1 \rangle$ = 0-module)
- $\Delta_{T(2,3)}(t) \sim t^2 - t + 1$

Fibered knots

Definition: $K \subset S^3$ is fibred if E_K fibres over S^1.

Prop: If E_K fibres over S^1 with monodromy $f: \Sigma \to \Sigma$, then $\tilde{E}_K \simeq \Sigma \times \mathbb{R}$. The action of the deck group is generated by the map $(x, t) \mapsto (\varphi(x), t+1)$.

Corollary: If E_K fibres over S^1 with monodromy $f: \Sigma \to \Sigma$, then

$$\Delta_K(t) \sim \det(tI - \varphi_*)$$

where $\varphi_*: H_1(\Sigma) \to H_1(\Sigma)$ is the homomorphism induced by the monodromy.

Corollary: If K is a fibered knot, then $\Delta_K(t)$ is monic and of degree $2g$, where g is the genus of the fibre.

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Definition: Suppose M is a module over a commutative ring R. Then M is finitely presented if there's an exact sequence

$$0 \to R^n \xrightarrow{P} R^m \xrightarrow{\pi} M \to 0$$

and this sequence is a presentation of M.

If $e_i = (0, \dots, 1, \dots 0) \in R^m$, then $\pi(e_1), \dots, \pi(e_m)$ are generators, and $P(e_1), \dots, P(e_n)$ are relations between the generators

m rows, n columns.

Write $P(e_j) = \sum_{i=1}^{m} P_{ij} e_i$. Say $m \times n$ matrix $[P_{ij}]$ is a presentation matrix for M.

Denote : generators $a_i := \pi(e_i)$, $\leftrightarrow$ rows
relations $r_j = P(e_j)$. $\leftrightarrow$ columns

Moves: add a generator a_{m+1} and relation $a_{m+1} = 0$ $\quad P \to \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$
add trivial relation $\quad P \longrightarrow (P \ 0)$
replace generator a_i with $a_i + \alpha a_j \to j^{th}$ row $= (j^{th}$ row of $P) - \alpha\, (i^{th}$ row of $P)$
replace relation r_i with $r_i + \beta r_j \to i^{th}$ col $= (i^{th}$ col of $P) + \beta\, (j^{th}$ col of $P)$
multiply row/col by units

R a UFD $\to \alpha_1, \dots, \alpha_k$, then $\gcd(\alpha_1, \dots, \alpha_k)$ well defined up to $\times$ by unit.

Dfn: $e_0(P) = \gcd(\{\det(\tilde{P})\})$, $\tilde{P}$ = $m \times m$ submatrix of P.

P and P' related by elementary move, $\Rightarrow$ $e_0(P) \sim e_0(P')$

Dfn: M a finitely presented module over a UFD, then define $e_0(M) = e_0(P)$.

Multivariable Alexander Polynomial

Dfn: universal abelian cover $p: \tilde{E}_L \to E_L$ is the connected covering space given by the kernel of the abelianisation map $|\cdot| : \pi_1(E_L) \to H_1(E_L) \cong \langle m_1, \dots, m_n \rangle \cong \mathbb{Z}^n$

$G_{Deck} \cong H_1(E_L) \cong \mathbb{Z}^n$, so $H_1(\tilde{E}_L)$ is a module over $\mathbb{Z}[H_1(E_L)] \cong \mathbb{Z}[\mathbb{Z}^n]$
$\cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] =: R_L$

R_L is a UFD

Dfn: the multivariable alexander polynomial $\Delta(L) = e_0(H_1(\tilde{E}_L)) \in R_L$, is well defined up to multiplication by a unit in R_L, i.e. $\pm t_1^{a_1} \dots t_n^{a_n}$.

Fox Calculus

X: 0-cell p, 1-cells $a_1, \dots, a_m$, n 2-cells $w_1, \dots, w_n$ along words in a_i's.

$\pi_1(X) = \langle a_1, \dots, a_m \mid w_1, \dots, w_n \rangle$, $H_1(X) = \mathbb{Z}^k \oplus T$. Let $\overline{H_1(X)} = H_1(X)/T \cong \mathbb{Z}^k$

abelianization map: $|\cdot| : \pi_1(X) \to H_1(X) \to \overline{H_1(X)} \cong \mathbb{Z}^k$ (surjective)

$p: \tilde{X} \to X$ correspond to $\ker|\cdot|$. Then $G_{Deck} \cong \mathbb{Z}^k$

$\hookrightarrow \tilde{X}$ has cell structure $g\tilde{e}$, $g \in G_{Deck}$.

Cells in $\tilde{X}$ are lifts of cells in X, $\Rightarrow \partial$ in $C_*^{cell}(X)$ commutes w/ action of G_{Deck},

$\Rightarrow C_*^{cell}(X)$ is a chain complex over $\mathbb{Z}[\overline{H_1(X)}] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$

Gives presentation for $H_1(\tilde{X})$:

$$C_*^{cell}(X): \quad \underset{\langle \tilde{w}_1, \dots, \tilde{w}_n \rangle}{R_X^n} \xrightarrow{d_2} \underset{\langle \tilde{a}_1, \dots, \tilde{a}_m \rangle}{R_X^m} \xrightarrow{d_1} \underset{\langle \tilde{p} \rangle}{R_X}$$

Boundary maps:

$d_1(\tilde{a}_i) = (|a_i| - 1)\tilde{p} \Rightarrow d_1 = [\,|a_1| - 1 \ \dots \ |a_m| - 1\,]$

$d_2: R_X^n \xrightarrow{A_X} R_X^m$, where $A_X = [d_{a_i} w_j]$, $d_{a_i} w_j$ is the *Fox derivative*

$$d_{a_i}\left(\prod_{k=1}^{r} a_{i_k}^{\pm 1}\right) = \sum_{k=1}^{r} |a_{i_1}^{\pm 1} \dots a_{i_{k-1}}^{\pm 1}| \, d_{a_i}(a_{i_k}^{\pm 1}) \quad \in \mathbb{Z}[\overline{H_1(X)}]$$

$$d_{a_i}(a_j) = \delta^i_j, \qquad d_{a_i}(a_j^{-1}) = \delta^i_j(-|a_j^{-1}|)$$

Lemma: $d_{a_i}(ww') = d_{a_i}(w) + |w| d_{a_i}(w')$

Group presentations: $G = \underbrace{\langle a_1, \dots, a_m \mid w_1, \dots, w_n \rangle}_{P}$ finitely presented group (presentation P)

associate cell complex X_P: 1-cells $\leftrightarrow a_i$, 2-cells $\leftrightarrow w_j$. $A_P = A_{X_P}$ alexander matrix

Tietze moves:

(1) add new generator a_{m+1} and new relation $a_{m+1} = 0$: $A_P \to \begin{bmatrix} A_P & 0 \\ 0 & 1 \end{bmatrix}$

(2) add trivial relation : $A_P \to [A_P \ 0]$

(3) multiply one relation by another: $P' = \langle a_1, \dots, a_m \mid w_1, \dots, w_i w_j, \dots, w_j, \dots, w_n \rangle$
: i^{th} col of A_P' = i^{th} + j^{th} col. of A_P

(4) replace w_j with $w_j' = a_i w_j a_i^{-1}$: multiply j^{th} col of A_P by $|a_i|$ (a unit)

Tietze: P and P' presentations of isomorphic groups, then we can get from P to P' by a sequence of Tietze moves.

Dfn: P a group presentation w/ m generators and n-relations, let

$\Delta(P) = e_1(A_P) = \gcd(\det \tilde{A} \mid \tilde{A}$ is an $(m-1)\times(m-1)$ submatrix of $A_P)$ [$n \times n$]

Prop: $(|a_j| - 1)\det A_{P,\hat{i}} = (|a_i| - 1)\det A_{P,\hat{j}}$.

Let $\alpha = \begin{cases} t - 1 & H_1(X_P) = \mathbb{Z} \\ 1 & H_1(X_P) \cong \mathbb{Z}^k \end{cases}$

Corollary: $(|a_i| - 1)\Delta(G) \sim \alpha \det(A_{P,\hat{i}})$

Summary of Calculating the Alexander polynomial of a link L:

(1) Calculate $\pi_1(E_K)$.

* Can calculate the multivariable Alexander polynomial $\Delta(L)$ by calculating the multivariable Alexander polynomial of $\Delta(\pi_1(E_K))$.

(2) Determine cell-structure of E_K, X, from $\pi_1(E_K)$.

(3) Define $p: \tilde{X} \to X$ the connected covering space of X associated to $\ker|\cdot| \leq \pi_1(X)$

(4) lift cell structure of X to one on $\tilde{X}$ via $G_{Deck} \cong \pi_1(X)/\ker|\cdot| \cong H_1(X) \cong \mathbb{Z}^k$

(5) Consider the cell chain complex as an R-module, $R = \mathbb{Z}[H_1(X)] \cong \mathbb{Z}\langle t_1^{\pm 1}, \dots, t_k^{\pm 1} \rangle$

(6) calculate $m \times n$ Alexander matrix A_X $(m \geq n)$ (representing d_2. can check $d_2 \circ d_1 = 0$)

(7) $\Delta_L(t) \sim \Delta(\pi_1(E_K)) = \gcd(\det\{\tilde{A}_X\})$, where $\tilde{A}_X$ is an $(m-1)\times(m-1) = n \times n$ submatrix of A_X

Rem: if L is a knot, then $\det(A_{P,\hat{m}}) \sim \Delta(E_K)$

Seifert genus

Dfn: $K \hookrightarrow S^3$ knot. $g(K) := \min\{g(S) : S$ is a seifert surface of $K\}$

Prop: $g(K) = 0 \iff K = U$

$\tilde{E}_K$ via Seifert surfaces

Lemma: ν_{S^3}/S trivial $\Rightarrow$ $\nu(S) \cong S \times [-1,1]$. Let $E_S = S^3 \setminus \mathrm{int}(\nu(S))$. Then

$\partial E_S = S \times \{-1\} \cup \partial S \times [-1,1] \cup S \times \{1\}$ $\Rightarrow \partial E_S$ has genus $2g(S)$. (E_S = double of S)

Lemma: E_S connected, and $H_1(E_S) \cong \mathbb{Z}^{2g(S)}$

Lemma: as a module over $R = \mathbb{Z}[H_1(E_K)] \cong \mathbb{Z}[t^{\pm 1}]$, $H_1(Y) \cong \mathrm{coker}(t i_{-*} - i_{+*})$,

where $i_{\pm *}$ are induced by inclusions $i_\pm : S \to S_\pm \subset \partial E_S$, i.e.

$i_{\pm *} : H_*(S) \to H_*(E_S)$

Thm (Seifert): if K is a knot, then $\deg(\Delta_K(t)) \leq 2g(K)$

Rem: if K alternating or ≤ 10 crossings, then $\deg(\Delta_K(t)) = 2g(K)$.

Fibred Knots:

E_K fibres over S^1 w/ connected fibre F

$\Rightarrow F$ a seifert surface for K (compare w/ second proof on existence).

Cor: $g(K) = g(F)$,

Cor: if K is fibred, then $\Delta_K(t)$ is monic

Statement is iff when K is alternating or ≤ 10 crossings